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# Extrapolating Structure Functions to Very Small $x$

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## Abstract

We review small  $x$  contributions to perturbative evolution equations for parton distributions, and their resummation. We emphasize in particular the resummation technique recently developed in order to deal with the apparent instability of naive small  $x$  evolution kernels and understand the empirical success of fixed-order perturbation theory. We give predictions for the gluon distribution and the structure functions  $F_2(x, Q^2)$  and  $F_L(x, Q^2)$  in an extended kinematic region, such as would be relevant for THERA or LEP+LHC  $ep$  colliders.

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Measurements of the inclusive structure functions  $F_2(x, Q^2)$  and  $F_L(x, Q^2)$  at HERA have shown that the scaling violations of structure functions are in extremely good agreement with the perturbative next-to-leading order (NLO) QCD prediction, down to the smallest values of  $x$ , and for all  $Q^2 \gtrsim 1 \text{ GeV}^2$  [1]. This agreement is surprising in that it is known that perturbative corrections beyond NLO in  $\alpha_s$  are enhanced by powers of  $\xi \equiv \ln(1/x)$ , and thus one would expect higher order corrections to be sizable whenever  $\alpha_s(Q^2)\xi \gtrsim 1$ , i.e. in most of the HERA kinematic region. Whereas techniques for the inclusion of small  $x$  contributions to leading twist evolution equations have been known for some time [2, 3], only recently did a consistent picture of the general structure of these contributions and their resummation emerge. Indeed, considerable theoretical progress has been spurred by the determination [4] of next-to-leading corrections to the BFKL kernel, which allows the computation of the next-to-leading  $\log(1/x)$  (NLLx) contributions to anomalous dimensions to all orders in  $\alpha_s$ . Specifically, it is now understood that the inclusion of NLLx contributions leads to instability [5] of perturbative evolution, unless it is suitably combined with a resummation of the collinear singularities [6–8] which are resummed order by order in the standard QCD evolution equations. Furthermore, the NLLx perturbative corrections give rise to increasingly large contributions to high orders of perturbation theory [9, 10] that make a nonsense of the perturbative expansion and call for an all-order resummation of the small- $x$  behaviour of the anomalous dimensions [11, 12].

Practical methods to deal with these issues have been developed recently [7, 13], and lead to a resummation prescription which is amenable to numerical treatment and direct comparison with the data. It then appears that the observed smallness of perturbative higher order corrections at small  $x$  can be accommodated within the current knowledge of the general structure of anomalous dimensions, but it poses very stringent constraints on the form of the unknown higher order terms. Furthermore, even when these constraints are respected, so that, as required by the data, deviations of the behaviour of the observable structure functions from the fixed next-to-leading order prediction are very small, still non-negligible modifications of the fitted parton distributions at small  $x$  are found. This, because of ambiguities in the resummation procedure, entails larger uncertainties on parton distributions at small  $x$ . Likewise, these corrections have a sizable impact on the extraction of  $\alpha_s$  from small  $x$  data, both on the central value and the estimates of overall theoretical uncertainties [13, 14].

In the wider kinematic region available at THERA the small differences between resummed and fixed-order predictions could be put to more stringent tests. This would allow one to pin down more precisely the ambiguities in the resummation procedure, thereby reducing the uncertainty on parton distributions at small  $x$  and on precision determinations of  $\alpha_s$  at small  $x$ . Also, the possibility of reaching smaller values of  $x$  for given  $Q^2$  would allow a test of resummed perturbation theory in a region where the relevant resummation parameter  $\alpha_s\xi$  is large, and also to see whether the perturbative description of scaling violations remains satisfactory or starts to break down, as is often suggested [15].

Here we briefly review our current understanding of resummed perturbation theory at small  $x$ . We then give predictions for the gluon distribution and the structure functions  $F_2$  and  $F_L$  in two different resummation scenarios, and compare these to fixed next-to-leading order results in the kinematic range which is relevant for THERA. This is

essentially the same kinematic region accessible at a hypothetical lepton–hadron collider obtained combining LEP with the LHC, so our predictions would also be relevant at such a machine.

## 1 Duality of small $x$ evolution

The basic result which allows the determination of contributions to anomalous dimensions which are logarithmically enhanced in  $x$  to all orders in the coupling, and thus their inclusion in evolution equations, is the *duality* of perturbative evolution [7, 16]: because leading–twist evolution of structure functions takes place both in  $x$  and  $Q^2$ , it admits a dual description in terms of equations for evolution in  $t = \ln(Q^2/\mu^2)$  or evolution in  $\xi = \ln(1/x)$ . This property is easy to prove [16] when the coupling is fixed, and can be shown to remain valid when the coupling runs by explicit order–by–order perturbative computation [12].

Let us first consider for simplicity the case (relevant in the very small  $x$  limit) of a single parton distribution  $G(\xi, t)$ , identified with the dominant eigenvector of perturbative evolution. The pair of dual evolution equations are then

$$\frac{d}{dt}G(\xi, t) = P(\xi, \alpha_s) \otimes G(\xi, t), \quad (1)$$

$$\frac{d}{d\xi}G(\xi, t) = K(t, \alpha_s) \otimes G(\xi, t), \quad (2)$$

The convolutions on the right–hand sides of the dual evolution equations (1–2) are with respect to  $\xi$  in the first equation ( $P(\xi, \alpha_s)$  is the usual splitting function) and with respect to  $t$  in the second equation;  $\alpha_s = \alpha_s(t)$  and is unaffected by convolutions. Duality means that the solutions to these equations coincide up to higher twist corrections provided the respective boundary conditions and kernels are suitably matched.

The detailed form of the matching of boundary conditions is irrelevant for our purposes, but it is important to notice that the matching is such that the boundary condition to (1) depends only on  $\xi$  (and not on  $t$ ) and the boundary condition to (2) depends only on  $t$  (and not on  $\xi$ ) as required by factorization. The matching of the kernels is given by the duality equation

$$\chi(\gamma(N, \alpha_s), \alpha_s) = N, \quad (3)$$

or equivalently its inverse

$$\gamma(\chi(M, \alpha_s), \alpha_s) = M. \quad (4)$$

Here  $\gamma$  is the usual anomalous dimension, related to the splitting function by Mellin transformation with respect to  $\xi$ :

$$\gamma(N, \alpha_s) = \int_0^\infty d\xi e^{-N\xi} P(\xi, \alpha_s). \quad (5)$$

The relation between  $\chi(M, \alpha_s)$  and  $K(t, \alpha_s)$  is somewhat more complicated because, upon Mellin transformation with respect to  $t$  the running coupling  $\alpha_s(t)$  on the right-hand side of Eq. (1) becomes a differential operator. The relation between the evolution kernel  $K(t, \alpha_s)$  and the dual kernel  $\chi(M, \alpha_s)$  can nevertheless be determined order by order in perturbation theory [12]: defining

$$K(t, \alpha_s) = \alpha_s K_0(t) + \alpha_s^2 K_1(t) + \dots, \quad (6)$$

$$\chi(M, \alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \dots, \quad (7)$$

we get

$$\begin{aligned} \chi_0(M) &= \int_{-\infty}^{\infty} dt e^{-Mt} K_0(t); \\ \chi_1(M) &= \int_{-\infty}^{\infty} dt e^{-Mt} K_1(t) + \frac{\beta_0}{4\pi} \frac{1}{2} \frac{\chi_0(M) \chi_0''(M)}{\chi_0'(M)^2}; \dots, \end{aligned} \quad (8)$$

where  $\beta_0 = \frac{11}{3}n_c - \frac{2}{3}n_f$  is the first coefficient of the QCD  $\beta$  function.

It follows from the form of the duality equation (3) that knowledge of the leading (next-to-leading, ...) term in the expansion of  $\chi$  in powers of  $\alpha_s$  at fixed  $M$  determines the leading (next-to-leading, ...) term in the expansion of  $\gamma$  in powers of  $\alpha_s$  at fixed  $\alpha_s/N$ : i.e. defining further

$$\gamma(N, \alpha_s) = \gamma_s\left(\frac{\alpha_s}{N}\right) + \alpha_s \gamma_{ss}\left(\frac{\alpha_s}{N}\right) + \dots, \quad (9)$$

then

$$\chi_0\left(\gamma_s\left(\frac{\alpha_s}{N}\right)\right) = \frac{N}{\alpha_s}, \quad \gamma_{ss}\left(\frac{\alpha_s}{N}\right) = -\frac{\chi_1\left(\gamma_s\left(\frac{\alpha_s}{N}\right)\right)}{\chi_0'\left(\gamma_s\left(\frac{\alpha_s}{N}\right)\right)}, \dots \quad (10)$$

Likewise, knowledge of the leading, next-to-leading, ... terms in the expansion of  $\gamma$  in powers of  $\alpha_s$  at fixed  $N$  determines the leading, next-to-leading, ... terms in the expansion of  $\chi$  in powers of  $\alpha_s$  at fixed  $\alpha_s/M$ : writing

$$\gamma(N, \alpha_s) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \dots, \quad (11)$$

$$\chi(M, \alpha_s) = \chi_s\left(\frac{\alpha_s}{M}\right) + \alpha_s \chi_{ss}\left(\frac{\alpha_s}{M}\right) + \dots, \quad (12)$$

then

$$\gamma_0\left(\chi_s\left(\frac{\alpha_s}{M}\right)\right) = \frac{M}{\alpha_s}, \quad \chi_{ss}\left(\frac{\alpha_s}{M}\right) = -\frac{\gamma_1\left(\chi_s\left(\frac{\alpha_s}{M}\right)\right)}{\gamma_0'\left(\chi_s\left(\frac{\alpha_s}{M}\right)\right)}, \dots \quad (13)$$

It should be understood that the running coupling corrections Eq. (8) are always included in the definition of  $\chi$  in the above equations.

Because the  $\xi$  evolution equation is essentially the same as the BFKL equation (up to factorization scheme and scale choices, which become relevant beyond leading order [4,17]) the duality relation can be viewed as a consistency condition between this equation and the standard renormalization group equation for moments of structure functions in the

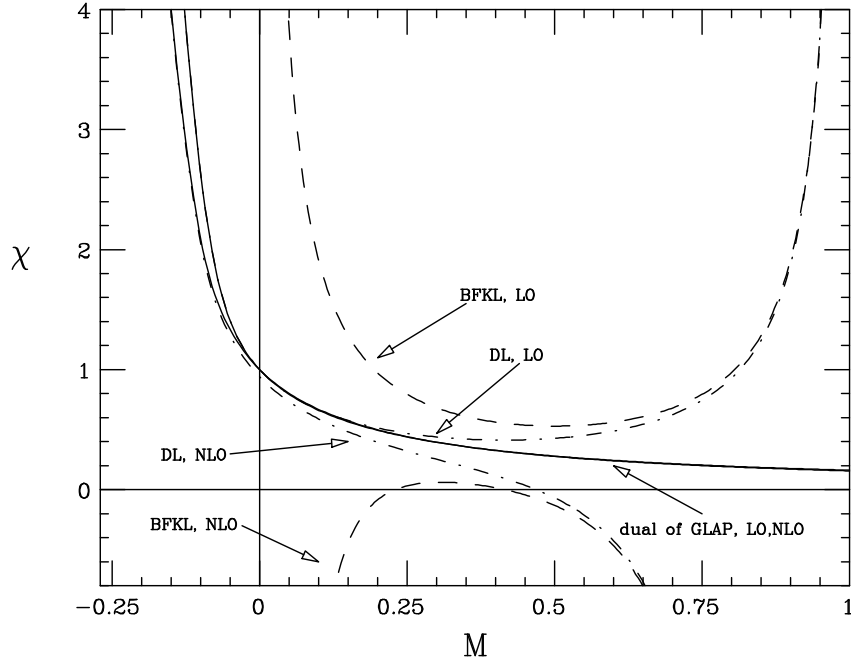


Figure 1: Plots of different approximations to  $\chi$ : the BFKL leading and next-to-leading order functions (7),  $\alpha_s \chi_0$  and  $\alpha_s \chi_0 + \alpha_s^2 \chi_1$  (dashed); the LO and NLO dual  $\alpha_s \chi_s$  and  $\alpha_s \chi_s + \alpha_s^2 \chi_{ss}$  (12) of the one and two loop anomalous dimensions (solid), and the double-leading functions at LO and NLO defined in Eq. (15) (dotdashed). All curves are computed with  $\alpha_s = 0.2$ .

region of their common validity (i.e. large  $Q^2$  and small  $x$ ). Hence, knowledge of the BFKL kernel  $K(t, \alpha_s)$  (6) can be translated into information of  $\chi$  (7), which in turn can be used to gain information on the logarithmically enhanced contributions  $\gamma_s, \gamma_{ss}, \dots$  (9) to the anomalous dimension  $\gamma(\alpha_s, N)$ , and conversely. In fact, the leading-order equation in (10) has been known for a long time [18]; the new insight here is that this is just a consequence of a more general duality.

## 2 The Double-Leading expansion

Only the first two orders in the expansion of  $\chi$  at fixed  $M$  and  $\gamma$  at fixed  $N$  are currently known. While the perturbative expansion of  $\gamma$  is well-behaved, in the sense that  $\alpha_s \gamma_1$  is a small correction to  $\gamma_0$  for reasonable values of the coupling constant, the perturbative expansion of  $\chi$  is very poorly behaved, in that the NLO correction  $\chi_1$  completely changes the qualitative shape of the kernel. In particular (see Fig. 1), in the physical region  $0 \leq M \leq 1$  the LO kernel has simple poles with positive residue at  $M = 0$  and  $M = 1$  and a minimum in between. The NLO correction  $\chi_1$  instead has higher order poles with negative coefficient, and, for any realistic value of  $\alpha_s$  (essentially, for all  $\alpha_s \gtrsim 0.03$ ) the full NLO function has just a maximum (for smaller  $\alpha_s$  it has a minimum and two maxima) [9].

It is easy to show that the solution to the evolution equation determined by a kernel with this shape displays unphysical oscillatory behaviour in the limit as  $x \rightarrow 0$ , and thus, in particular, leads to negative cross-sections [5].

Because the Mellin transform (8) of  $t^k = \ln^k(Q^2/\mu^2)$  is  $(k-1)!/M^{k-1}$ , the presence of  $1/M$  poles in the kernel  $\chi$  is related to collinear singularities: indeed, according to Eq. (4) the coefficients of these singularities are determined by knowledge of the anomalous dimensions  $\gamma_0, \gamma_1, \dots$  in the usual renormalization group equations, which resum collinear singularities. It is easy to understand [7] why these singularities lead to a series of poles in  $M = 0$  with alternating signs. Indeed, recall that momentum conservation implies that the largest eigenvalue of the anomalous dimension matrix vanishes at  $N = 1$ , i.e.  $\gamma(1, \alpha_s) = 0$ , which by duality (3) implies  $\chi(0, \alpha_s) = 1$ . It follows that if, in the vicinity of  $M = 0$ ,  $\chi_s$  behaves as

$$\chi_s \underset{M \rightarrow 0}{\sim} \frac{\alpha_s}{\alpha_s + M} = \frac{\alpha_s}{M} - \frac{\alpha_s^2}{M^2} + \frac{\alpha_s^3}{M^3} + \dots \quad : \quad (14)$$

the series of poles in  $\chi_s, \chi_{ss}, \dots$  actually sums up to the regular behaviour  $\chi(0, \alpha_s) = 1$ .

The poles in  $\chi$  as  $M \rightarrow 0$  are summed to all orders into  $\chi_s, \chi_{ss}, \dots$ , and thus the undesirable behaviour of the expansion of  $\chi$  can be removed by defining order by order an improved expansion. Namely, we define a *double leading expansion* where to each order in  $\alpha_s$  both the terms present in the expansion in powers of  $\alpha_s$  at fixed  $M$  and at fixed  $\alpha_s/M$  are included:

$$\begin{aligned} \chi(M, \alpha_s) = & \left[ \alpha_s \chi_0(M) + \chi_s \left( \frac{\alpha_s}{M} \right) - \frac{n_c \alpha_s}{\pi M} \right] \\ & + \alpha_s \left[ \alpha_s \chi_1(M) + \chi_{ss} \left( \frac{\alpha_s}{M} \right) - \alpha_s \left( \frac{f_2}{M^2} + \frac{f_1}{M} \right) - f_0 \right] + \dots \end{aligned} \quad (15)$$

In this expansion, in the vicinity of  $M = 0$  the singularities of  $\chi_0, \chi_1, \dots$  are resummed into  $\chi_s, \chi_{ss}$ , while the subtraction terms avoid double-counting of these contributions. Note (see Fig. 1) that at larger values of  $M$  the shape of  $\chi_0, \chi_1, \dots$  is reproduced, but in most of the  $M$  range the kernel Eq. (15) coincides with the (dual of) the standard anomalous dimensions  $\gamma_0$  and  $\gamma_1$ , consistent with the empirical smallness of small- $x$  correction to perturbative evolution. This also has the significant implication that the double-leading expansion of  $\chi$  is as stable as the usual expansion of  $\gamma$  at fixed  $N$ .

It is easy to show that the corresponding double leading expansion of  $\gamma$ ,

$$\begin{aligned} \gamma(N, \alpha_s) = & \left[ \alpha_s \gamma_0(N) + \gamma_s \left( \frac{\alpha_s}{N} \right) - \frac{n_c \alpha_s}{\pi N} \right] \\ & + \alpha_s \left[ \alpha_s \gamma_1(N) + \gamma_{ss} \left( \frac{\alpha_s}{N} \right) - \alpha_s \left( \frac{e_2}{N^2} + \frac{e_1}{N} \right) - e_0 \right] + \dots, \end{aligned} \quad (16)$$

is consistent with duality, in that  $\chi$  (15) and  $\gamma$  (16) are dual to each other order by order in the double-leading expansion, up to higher order corrections. Hence, for practical applications we may directly use the double-leading anomalous dimension (16) in the usual evolution equation (1). This will ensure that collinear singularities are resummed according to the renormalization group in the usual way, while leading logs of  $1/x$  are consistently included up to next-to-leading order.

For actual phenomenology, the full set of anomalous dimensions and coefficient functions are needed. It is easy to see that the double-leading expansion is consistent with

diagonalization of the anomalous dimension matrix, in the sense that one may equivalently, up to subleading corrections, construct a two by two matrix of double-leading anomalous dimensions and diagonalize it, or else construct directly a double-leading expansion of eigenvalues and projectors. Because one of the two eigenvectors of  $\gamma$  is free of small- $x$  singularities, so its double-leading expansion coincides with the standard expansion at fixed  $N$ , the latter procedure is in practice simpler. Hence, the double leading expansion can be fully defined in terms of the expansion of the large anomalous dimension eigenvalue, and of the quark-sector matrix elements which determine the projectors on the eigenvectors. Likewise, one can construct double-leading coefficient functions, and prove that the expansion transforms consistently upon changes of factorization scheme. Detailed proofs and results needed for a practical implementation are given in ref. [13].

### 3 Resummation

Even though the difference between the double-leading expansions of  $\chi$  (15) and  $\gamma$  (16) is subleading, it can in practice be large when  $M \gtrsim 0.25$ . Indeed, recall that duality (3) implies  $\chi = N$ . It is clear from Fig. 1 that in the region  $M \gtrsim 0.25$  the difference between the leading order and next-to-leading order double-leading curves is small for any fixed value of  $M$ , but it is quite large for a fixed value of  $\chi = N$ , because the curves are almost parallel to the  $M$ -axis: the LO BFKL curve has a minimum at  $M = 1/2$ . Since  $\gamma$  is a function of  $N$ , in this region the perturbative solution (10) of the duality relation (3) is not good and the expansion of  $\gamma$  is not well behaved.

A possible way out is to determine the double-leading  $\gamma$  (16) from the double-leading  $\chi$  (15) by solving the duality relation (3) exactly (rather than perturbatively). This can be done for instance by numerical methods, or equivalently by differentiating with respect to  $t$  the solution of the evolution equation (2) determined using the double-leading  $\chi$  kernel (15), as in ref. [8]. However, this approach, besides being cumbersome to implement in standard evolution codes, has the shortcoming that it hides a genuine perturbative ambiguity. Indeed, in this way the perturbative expansion of  $\gamma$  is in practice stabilized by assuming that in the region  $M \approx 1/2$  the (large) subleading corrections to  $\gamma$  will be such as to reproduce the shape of  $\chi$ , as computed to some fixed perturbative order, or possibly further improved according to a model of its behaviour at large  $M \sim 1$  [8].

We instead prefer to use only the available perturbative information on  $\gamma$ , without making model-dependent assumptions. It can be shown [12] that the poor perturbative behavior of the expansion of  $\gamma$  at fixed  $\alpha_s/N$  manifests itself in a rise of the associate splitting functions:  $P_{ss}/P_s \underset{\xi \rightarrow \infty}{\sim} \alpha_s \xi$ ,  $P_{sss}/P_s \underset{\xi \rightarrow \infty}{\sim} \alpha_s^2 \xi^2$  and so on. This rise can be removed by simply subtracting at each order a suitable constant  $c_i$  from  $\chi_i$  (computable order by order in perturbation theory as a function of  $\chi_i$  and their derivatives at  $M = 1/2$ ), and then determining  $\gamma_{ss\dots}$  from the subtracted  $\chi_i$ . Thus, the expansion of  $\gamma$  (7) can be stabilized by just reorganizing the perturbative expansion of  $\chi$ :

$$\chi(M, \alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \dots \quad (17)$$

$$= \alpha_s \tilde{\chi}_0(M) + \alpha_s^2 \tilde{\chi}_1(M) + \dots, \quad (18)$$

where

$$\alpha_s \tilde{\chi}_0(M, \alpha_s) \equiv \alpha_s \chi_0(M) + \Delta\lambda, \quad \tilde{\chi}_i(M) \equiv \chi_i(M) - c_i, \quad (19)$$

for  $i = 1, 2, \dots$ , and thus

$$\Delta\lambda \equiv \sum_{n=1}^{\infty} \alpha_s^{n+1} c_n. \quad (20)$$

If  $\chi$  has a minimum, then its value at the minimum coincides [7] with the value of  $\tilde{\chi}_0$  at its minimum  $M = 1/2$ , namely

$$\lambda \equiv \tilde{\chi}_0(\tfrac{1}{2}) = \chi_0(\tfrac{1}{2}) + \Delta\lambda. \quad (21)$$

Since the value of  $\chi$  at its minimum determines the asymptotic behaviour of the structure function as  $x \rightarrow 0$ , this implies that in order to remove the perturbative instability it is necessary and sufficient to resum the asymptotic small  $x$  behaviour into the leading order kernel  $\tilde{\chi}_0$ . The perturbative instability signals the fact that the all-order asymptotic behaviour must be known to all orders.

Of course, we are free to use any particular truncation of  $\Delta\lambda$  (20): for instance, we could simply take  $\chi$  to coincide with its NLO form in the double-leading expansion. Eq. (18) then provides us with a stable perturbative expansion of  $\gamma$ , which at NLO is very close to the exact dual of  $\chi$ , the large subleading corrections having been resummed in a minimal way. In this way Eq. (18) gives us a simple prescription which completely stabilizes the double-leading expansion of  $\gamma$  whenever the double-leading expansion of  $\chi$  is also stable. Hence, any specific resummation of  $\chi$  (such as that constructed in ref. [8]) can be accommodated in this formalism. Since however we prefer not to rely on such specific assumptions, we will consider  $\lambda$  (21) as a free parameter.

To NLO, the (resummed) expansion of  $\gamma$  obtained from Eq. (18) is related to the unresummed expansion obtained from Eq. (17) by

$$\tilde{\gamma}(N, \alpha_s) = \tilde{\gamma}_s\left(\frac{\alpha_s}{N}\right) + \alpha_s \tilde{\gamma}_{ss}\left(\frac{\alpha_s}{N}\right) + \dots, \quad (22)$$

where

$$\begin{aligned} \tilde{\gamma}_s\left(\frac{\alpha_s}{N}\right) &= \gamma_s\left(\frac{\alpha_s}{N-\Delta\lambda}\right), \\ \tilde{\gamma}_{ss}\left(\frac{\alpha_s}{N}\right) &= \gamma_{ss}\left(\frac{\alpha_s}{N-\Delta\lambda}\right) - \frac{\chi_1(\tfrac{1}{2})}{\chi'_0\left(\gamma_s\left(\frac{\alpha_s}{N-\Delta\lambda}\right)\right)}. \end{aligned} \quad (23)$$

Since the resummation only involves formally subleading terms,

$$\gamma_s + \alpha_s \gamma_{ss} = \tilde{\gamma}_s + \alpha_s \tilde{\gamma}_{ss} + O(\alpha_s^3/N). \quad (24)$$

A resummed double-leading expansion can finally be constructed by combining the resummed anomalous dimension  $\tilde{\gamma}$  (22) with the standard expansion of  $\gamma$  at fixed  $N$ . This gives a resummed double-leading expression for the large anomalous dimension eigenvector. It can further be shown [13] that resummed double-leading expressions for the full



matrix of anomalous dimensions and for coefficient functions can be obtained by performing the replacement  $N \rightarrow N - \Delta\lambda$  in all remaining quantities, i.e. the projectors and the coefficient functions.

The construction of the resummed double-leading expansion entails a further ambiguity in the treatment of the double counting subtractions in Eq. (16): because these terms are common to the fixed- $N$  expansion  $\gamma_0, \gamma_1, \dots$  and the fixed  $\alpha_s/N$  expansion  $\gamma_s, \gamma_{ss}, \dots$ , we are free to decide whether to leave them unaffected by the replacement  $N \rightarrow N - \Delta\lambda$  or not. This defines a pair of resummation procedures, which of course only differ by subleading terms. Clearly, a variety of intermediate alternatives would also be possible. The main difference between these prescriptions is the nature of the small  $N$  singularities of the anomalous dimension, which control the asymptotic small  $x$  behaviour. The resummed anomalous dimension always has a cut starting at  $N = \lambda$  Eq. (21), which corresponds [13] to an  $x^{-\lambda}$  behaviour of splitting functions at small  $x$ . If the subtractions are affected by the replacement, then  $\gamma_0$  and  $\gamma_1$  are the same as in the unresummed case (S-resummation), i.e. they have a simple pole at  $N = 0$ , which leads to a “double-scaling” [19] rise at small  $x$ . If the subtractions are unaffected, this pole is removed by the subtraction itself (R-resummation). It follows that if  $\lambda$  is positive, then the two resummations give similar results at small  $x$ , namely an  $x^{-\lambda}$  power rise. If  $\lambda \leq 0$ , the S-resummation will display double scaling at small  $x$ , while the R-resummation will display a valence-like  $x^{-\lambda}$  behaviour.

## 4 Predictions for THERA

A comparison of the resummation discussed in the previous sections to recent HERA data [20] was presented in ref. [13]. The best-fit results of that reference can be used to obtain predictions for THERA and discuss the study of small  $x$  scaling violations at such a facility. Because of the larger center-of-mass energy available at THERA, these predictions essentially amount to an extension of the current kinematic range of  $1/x$  by about a decade for each value of  $Q^2$ . It is interesting to note that more or less the same center-of-mass energy would be available at a hypothetical lepton-hadron collider obtained combining LEP with the LHC.

The phenomenological analysis of ref. [13] is based on a fit to data for the reduced cross-section

$$\sigma_{red}(x, y, Q^2) = F_2(x, Q^2) + \frac{y^2}{2(1-y) + y^2} F_L(x, Q^2). \quad (25)$$

determined from structure functions  $F_2$  and  $F_L$  computed to next-to-leading order in the double leading expansion with the R- and S-resummation prescriptions discussed in Sect. 3, by evolving parton distributions given at a scale  $Q_0 = 2$  GeV. A standard unresummed next-to-leading order fit is also performed for comparison. The fits are performed in the parton scheme, so the quark distribution coincides by construction with  $F_2$ . The large- $x$  shape of parton distributions is taken from a global fit, while the small  $x$  behaviour is parametrized by two free parameters  $\lambda_q$  and  $\lambda_g$ , which give the asymptotic

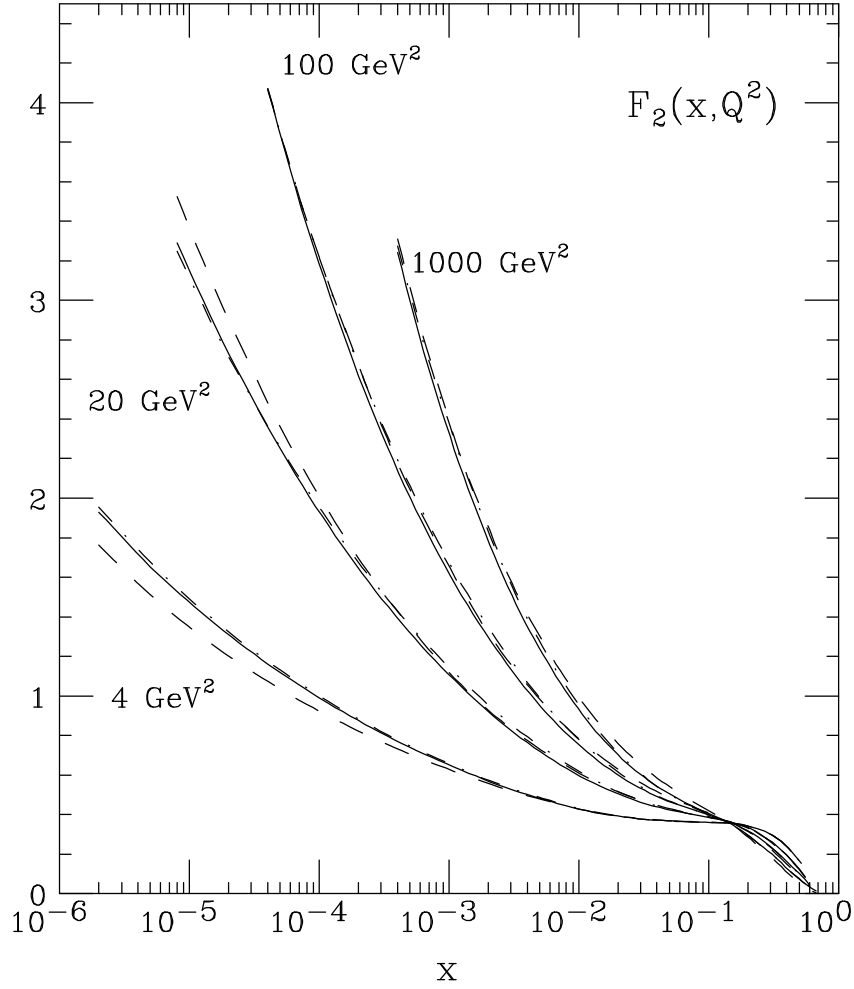


Figure 2: The structure function  $F_2(x, Q^2)$  obtained from a fit [13] to HERA data [20]. The prediction for THERA is the last decade in  $x$  for each value of  $Q^2$ . The solid curve is an unresummed fixed-order two loop fit, while the dot-dashed curve corresponds to the S-resummation and the dashed curve to the R-resummation discussed in Sect. 3.

small- $x$  behaviour of the singlet quark and gluon distributions respectively as  $x^{-\lambda_q}$ ,  $x^{-\lambda_g}$ . The resummation parameter  $\lambda$  Eq. (21) is also left as a free parameter. The strong coupling is fixed at  $\alpha_s(M_z) = 0.119$ .

Because at the initial scale  $Q_0$  abundant data are available down to the smallest values of  $x$ , and  $F_2$  coincides with the quark distribution, the quark exponent  $\lambda_q$  turns out to be the same in all fits, and gives the effective power rise of the  $F_2(x, Q_0^2) \sim x^{-\lambda_q}$ :  $\lambda_q \approx 0.2$ . The best-fit value of the gluon exponent is valence-like in all fits: in the two-loop fit it is  $\lambda_g \approx -0.1$ ; while in the resummed fits it is significantly more valence-like,  $\lambda_g \approx -0.2$  for both S- and R-resummation. The value of the resummation parameter  $\lambda$  instead varies significantly according to the resummation prescription which is adopted. For the S-resummation, any value  $\lambda \leq 0$  gives a good fit, with the best fit around  $\lambda \approx -0.25$ .

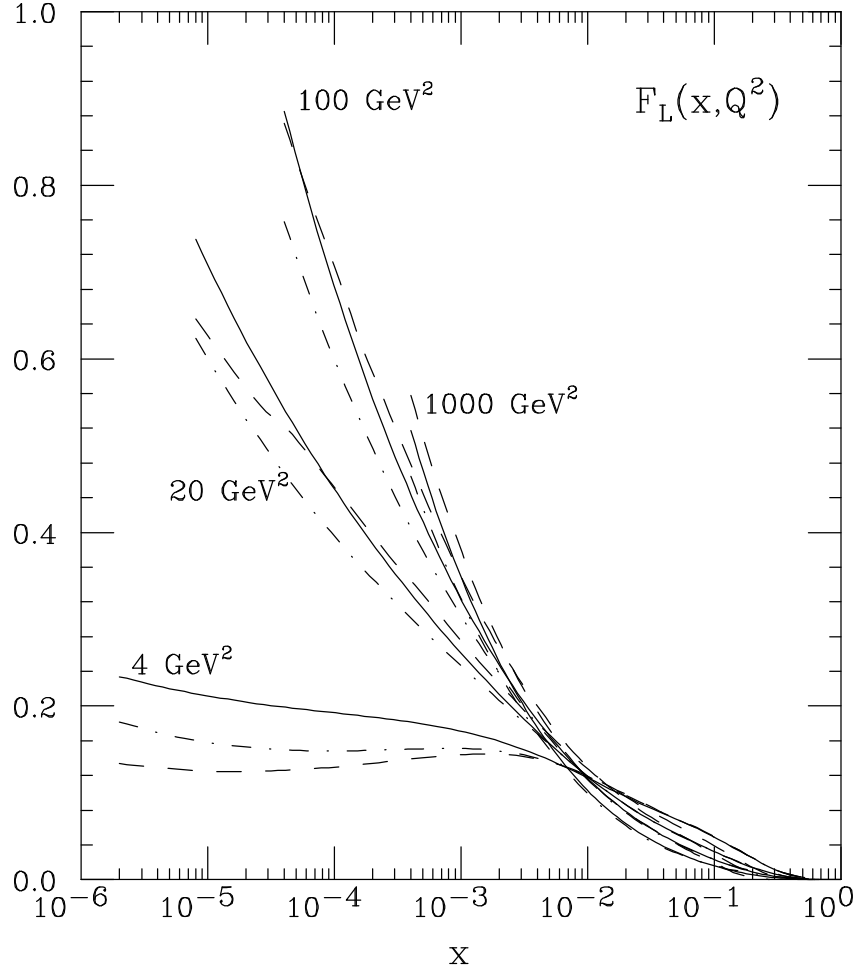


Figure 3: Same as Fig. 2, but for the structure function  $F_L(x, Q^2)$ .

As discussed in Sect. 3 the S-resummation with vanishing or negative  $\lambda$  is closest to the unresummed fixed-order result. With the R-resummation, instead, only a fine-tuned value of  $\lambda \approx 0.2$  gives a good fit. This value of  $\lambda$  turns out to be the same which one gets by fine-tuning the resummed anomalous dimension so that it be closest to the unresummed one in the HERA kinematic region [7]. Both resummed fits give a similar  $\chi^2 \approx 52$  with 93 degrees of freedom, to be compared to the unresummed value  $\chi^2 = 60$ .

The structure function  $F_2(x, Q^2)$  obtained in these fits is displayed in Fig. 2. It is apparent that, given the high precision of the HERA data, all curves, which give good fits to the data, are constrained to lie essentially on top of each other throughout the HERA region, except possibly at the smallest  $x$  values  $x \lesssim 10^{-4}$  at the initial scale  $Q_0$ , where the R-resummation curve rises slightly less. In the THERA range it is still very difficult to tell the difference between various prescriptions at higher scales  $Q^2 \geq 100 \text{ GeV}^2$ , but at lower scales, while results in the S-resummation are still essentially indistinguishable from the two-loop ones, the R-resummation predicts a somewhat faster evolution.

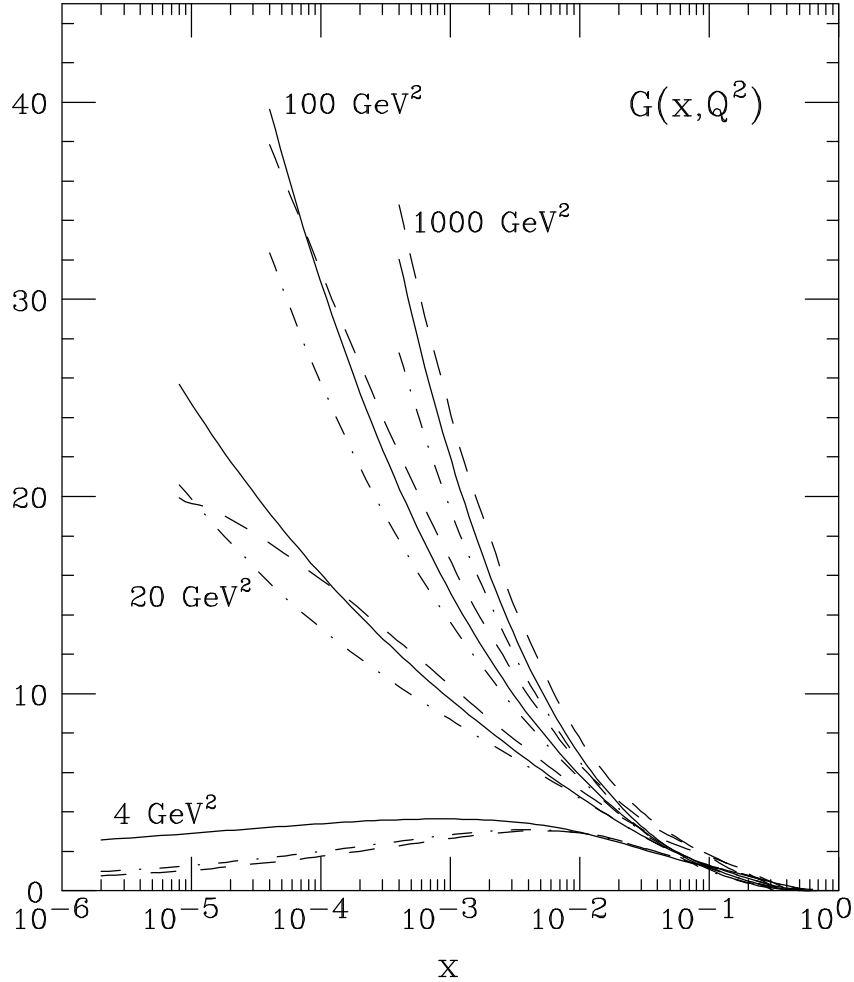


Figure 4: Same as Fig. 2, but for the  $\overline{\text{MS}}$  gluon distribution  $G(x, Q^2) = xg(x, Q^2)$ .

The structure function  $F_L$  is displayed in Fig. 3. This structure function is not determined very accurately by the HERA data for the reduced cross section Eq. (25), essentially because of the scarcity of large- $y$  data. The spread of the results is accordingly larger. Because  $F_L$  at small  $x$  has a large gluonic component, the behaviour of  $F_L$  is similar to that of the gluon distribution, displayed in Fig. 4. For ease of comparison with other work, the  $\overline{\text{MS}}$  gluon is shown, even though our fits were performed in the DIS scheme. Both resummations give a rather softer behaviour than the fixed-order one at the initial scale  $Q_0$ : valence-like for the gluon, turned into a rise of  $F_L$  at very small  $x$  (well into the THERA range) by the rise of the coefficient function. The R-resummation, however, then leads to significantly more rapid evolution: as the scale increases, the resummed gluon overtakes the fixed-order one. This is essentially due to the fact that the R-resummation eventually generates an  $x^{-\lambda}$  power behaviour of all parton distributions at small  $x$ , and here  $\lambda = 0.2$  (power rise). The S-resummation, instead, leads to evolution dominated by double scaling, which is very similar to the fixed-order one, and thus both the gluon and

$F_L$  preserve the relative softness that they displayed at the initial scale.

Summarizing, it is clear that resummation effects, though very small, become increasingly important as  $x$  decreases. Deviations from the fixed-order behaviour appear, at least in a simultaneous determination of  $F_2$  and  $F_L$ . At present, the deviations from the fixed order prediction are within the uncertainties of the resummation procedure: so, while it is clear that the resummed gluon distribution is softer than the unresummed one, it is hard to tell whether it will evolve faster or slower at small  $x$ . The underlying physics between these options is quite different: either the onset of a slow power-like rise (R-resummation), or persistence of the double-scaling rise (S-resummation). Both possibilities are consistent with present-day data, as well as with our current knowledge of anomalous dimensions. Understanding which (if any) of these possibilities is correct could be of considerable theoretical interest, and in particular, it could shed light on the running of the coupling in the high-energy limit [13, 16].

In conclusion, accurate data in the THERA region could reveal significant differences between the resummations procedures, and thus shed light on the structure of unknown higher order contributions to perturbative anomalous dimensions, and on the underlying physics. The simultaneous measurement of  $F_2$  and  $F_L$  in a wide range of  $Q^2$  at small  $x$  would allow an accurate determination of structure functions at small  $x$  which are required e.g. for precise phenomenology of heavy quark production at future colliders.

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## References

- [1] A. De Roeck, Nucl. Phys. **A666**, 129 (2000).
- [2] R. D. Ball and S. Forte, Phys. Lett. **B351**, 313 (1995).
- [3] R. K. Ellis, F. Hautmann and B. R. Webber, Phys. Lett. **B348**, 582 (1995).
- [4] V. S. Fadin and L. N. Lipatov, Phys. Lett. **B429**, 127 (1998).
- [5] D. A. Ross, Phys. Lett. **B431**, 161 (1998).
- [6] G. P. Salam, JHEP **07**, 019 (1998).
- [7] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. **B575**, 313 (2000).
- [8] M. Ciafaloni, D. Colferai and G. P. Salam, Phys. Rev. **D60**, 114036 (1999).
- [9] R. D. Ball and S. Forte, *Corrections at small  $x$* . Preprint hep-ph/9805315, 1998.
- [10] J. Blumlein, V. Ravindran, W. L. van Neerven and A. Vogt, *The unpolarized gluon anomalous dimension at small  $x$* . Preprint hep-ph/9806368, 1998.

- [11] Y. V. Kovchegov and A. H. Mueller, Phys. Lett. **B439**, 428 (1998).
- [12] R. D. Ball and S. Forte, Phys. Lett. **B465**, 271 (1999).
- [13] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. **B599**, 383 (2001).
- [14] R. D. Ball and S. Forte, Phys. Lett. **B358**, 365 (1995);  
R. D. Ball and S. Forte, *Inclusive measurement of the strong coupling at HERA*. Preprint hep-ph/9607289, 1996.
- [15] A. H. Mueller, *Small- $x$  physics, high parton densities and parton saturation in QCD*. Preprint hep-ph/9911289, 1999;  
Raju Venugopalan, *Deeply inelastic scattering off nuclei at RHIC*. Preprint hep-ph/0102087, 2000.
- [16] R. D. Ball and S. Forte, Phys. Lett. **B405**, 317 (1997).
- [17] M. Ciafaloni and G. Camici, Phys. Lett. **B430**, 349 (1998).
- [18] T. Jaroszewicz, Phys. Lett. **B116**, 291 (1982).
- [19] A. De Rújula et al., Phys. Rev. **10**, 1649 (1974);  
R.D. Ball and S. Forte, Phys. Lett. **B335**, 77 (1994).
- [20] H1 Coll., C. Adloff et al., *Deep-inelastic inclusive ep scattering at low  $x$  and a determination of  $\alpha_s$* . Preprint hep-ex/0012053, 2000.